Day 1
Lecture 2:
Basic concepts of compartmental models

Short course on modelling infectious disease dynamics in R

Ankara, Türkiye, September 2025

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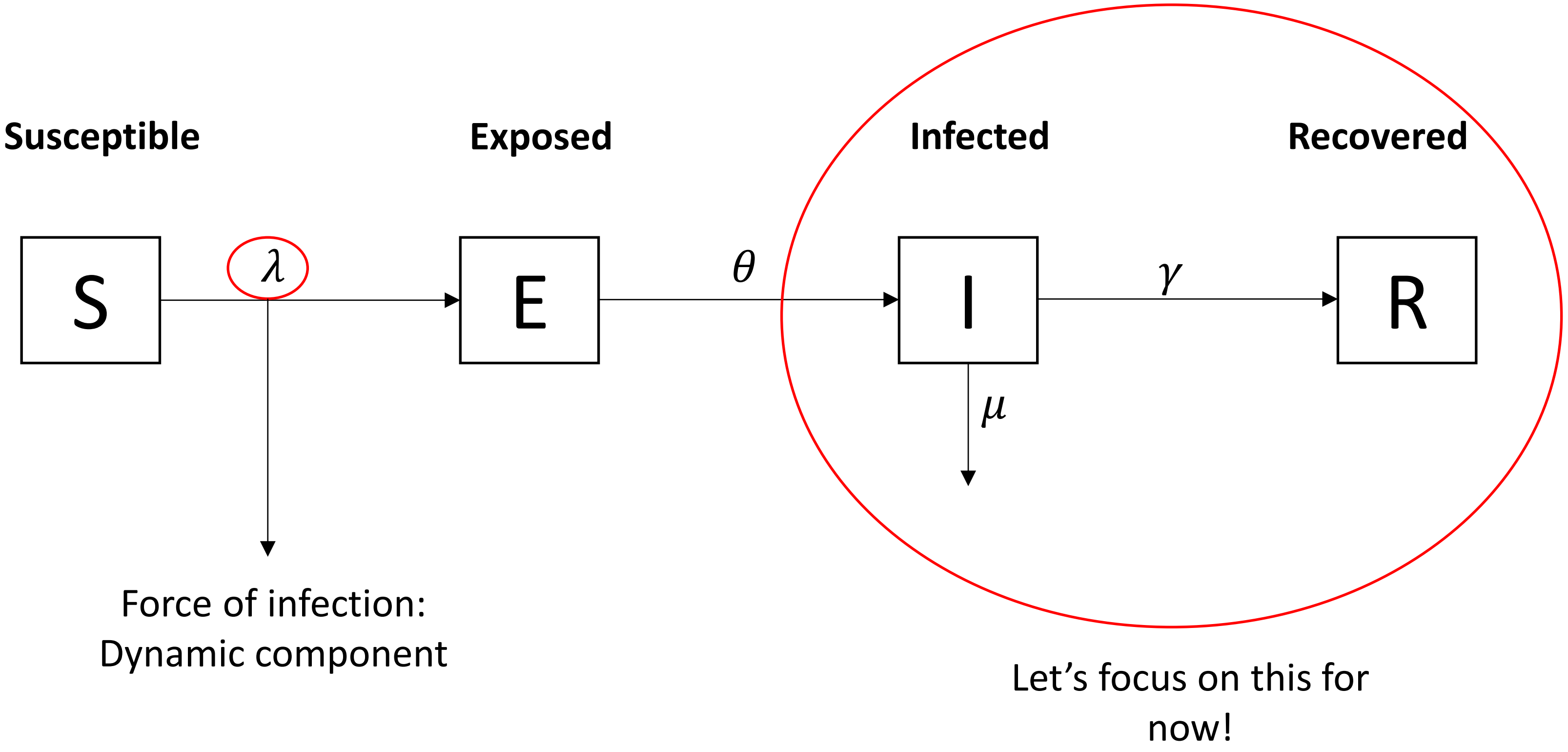
Aims of the session

- To Understand how disease models are designed
- Review concepts of probabilities, rates and competing hazards
- Understand the assumptions behind compartmental models

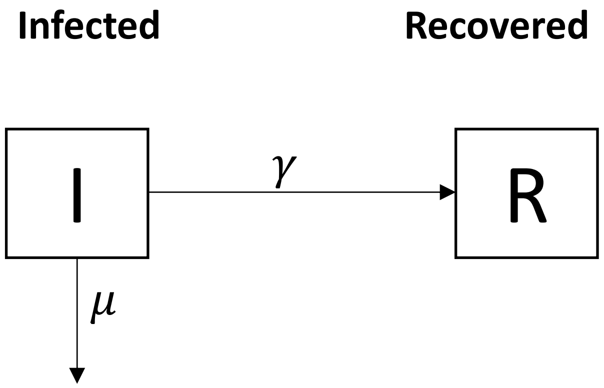
Checklist for designing a disease model

1) Describe the natural course of disease
2) Identify the necessary transitions between compartments
3) Interpret these transitions and find the relevant data to estimate the parameters
4) Review your research question and adapt your model complexity accordingly
5) Code your model!

A simple example

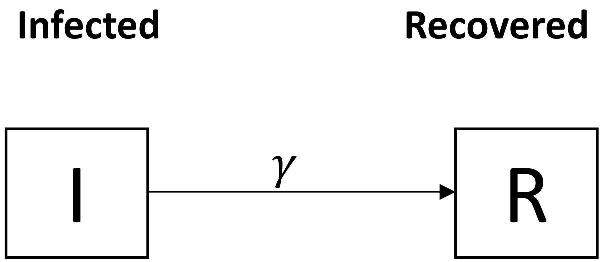


A cohort model



- How do we specify rates of transitions?
- How multiple rates affect a compartment?

A cohort model

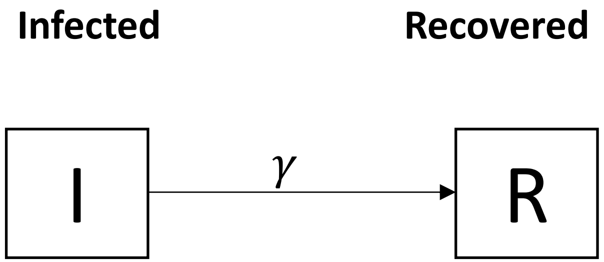


$$\frac{dI}{dt} = -\gamma I(t)$$

$$\frac{dR}{dt} = \gamma I(t)$$

- The rate of flow out of compartment *I* is proportional to the number of people on *I*
- γ is the proportionality constant
- We call γ a **constant hazard**
- A hazard is the event rate at a specific time t

A cohort model

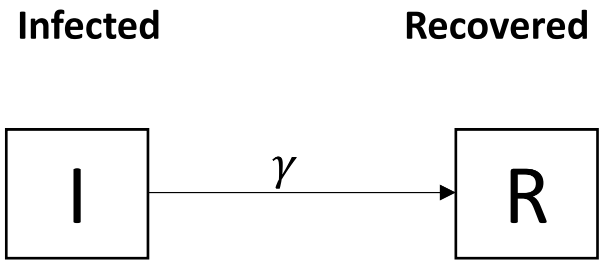


$$\frac{dI}{dt} = -\gamma I(t)$$

$$\frac{dR}{dt} = \gamma I(t)$$

- In our cohort γ is the recovery rate
- The larger γ is, the quicker they recover
- This means γ must be expressed in units of inverse time day^{-1}
- An average recovery rate of 10 days = 0.1 day^{-1} =1/10

So how do we interpret this transitions?



$$\frac{dI}{dt} = -\gamma I(t)$$

$$\frac{dR}{dt} = \gamma I(t)$$

- An initial cohort of 1000 people infected
- A recovery rate of 0.1 day^{-1} (10 days)
- When will 50% of infected be recovered?

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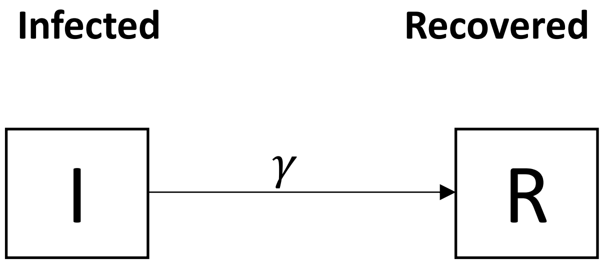

Solution

$$\frac{dI}{dt} = -\gamma I(t) \Longrightarrow I = I_0 e^{-\gamma t}$$

$$\frac{dR}{dt} = \gamma I(t) \Longrightarrow R = I_0(1 - e^{-\gamma t})$$

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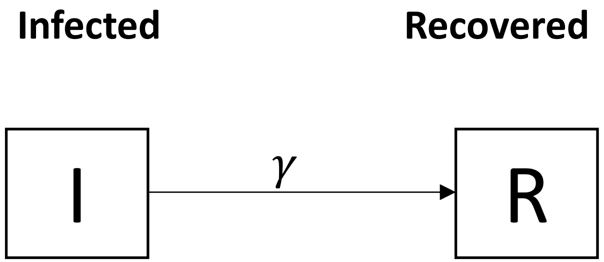

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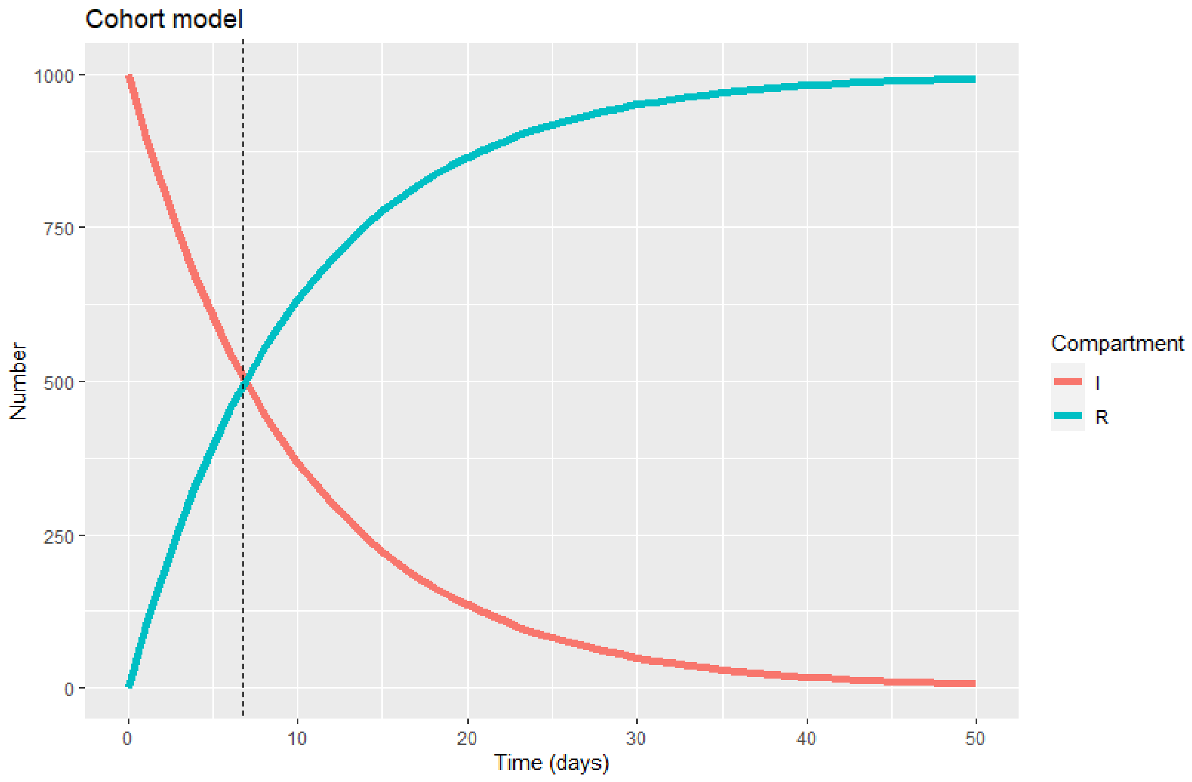
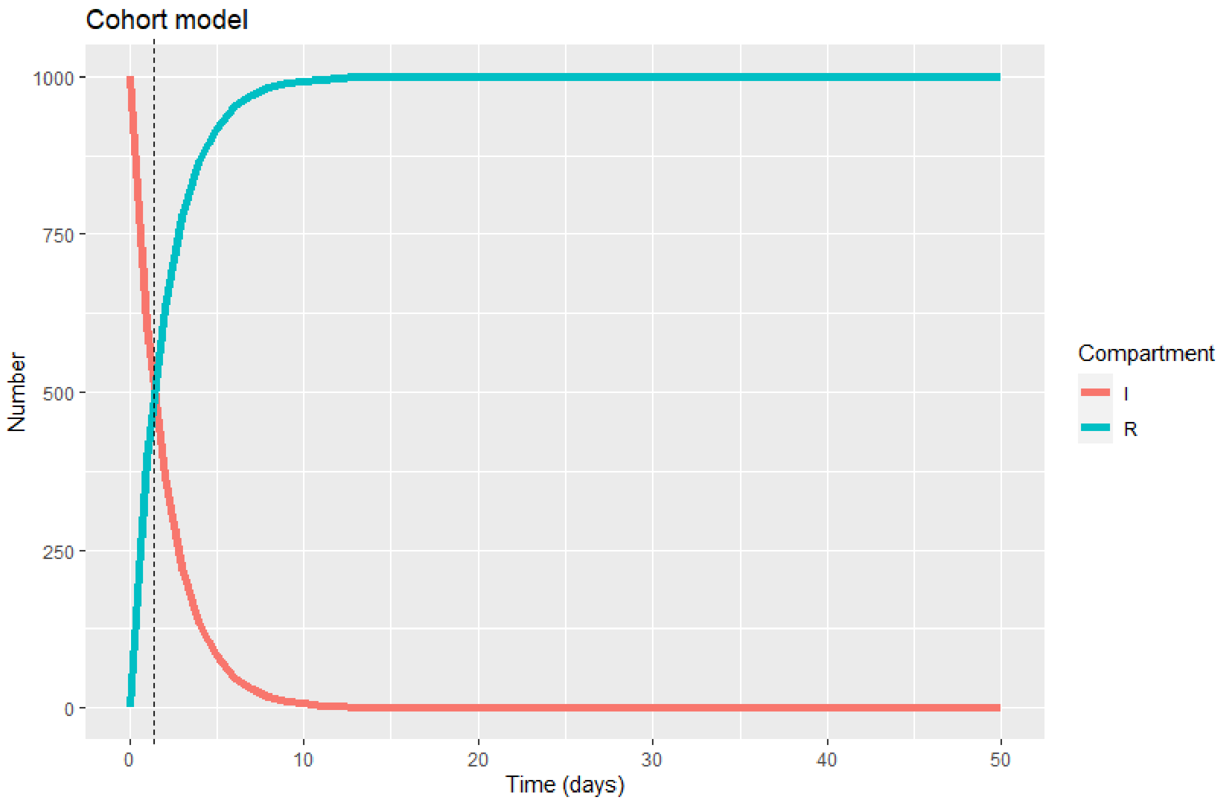
$$\frac{dR}{dt} = \gamma I(t) \quad \Longrightarrow \quad R = 1000(1 - e^{-0.1(10)})$$

So how do we interpret this transitions?

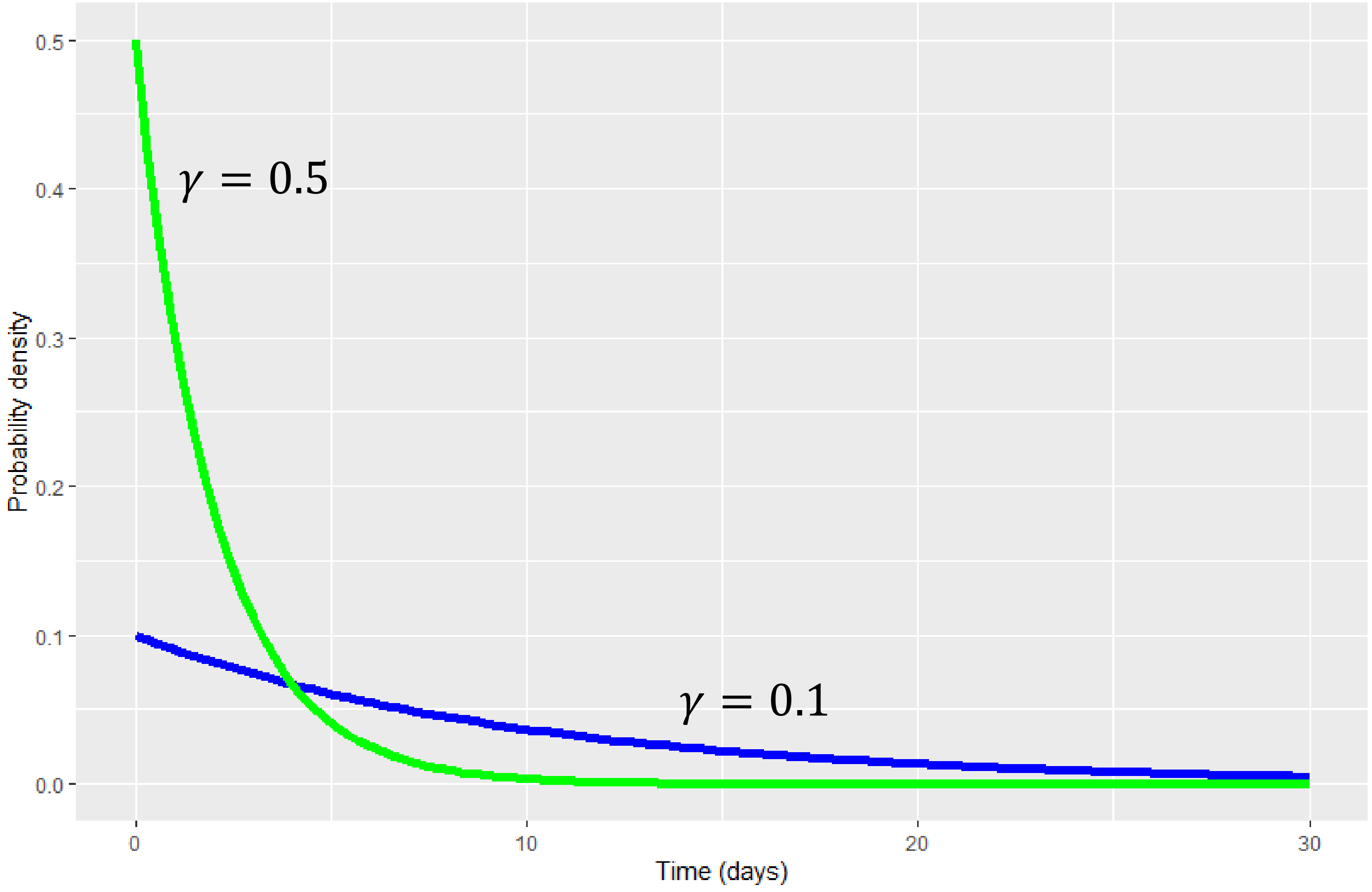
Recovery rate of 10 days



Recovery rate of 2 days. Much quicker !

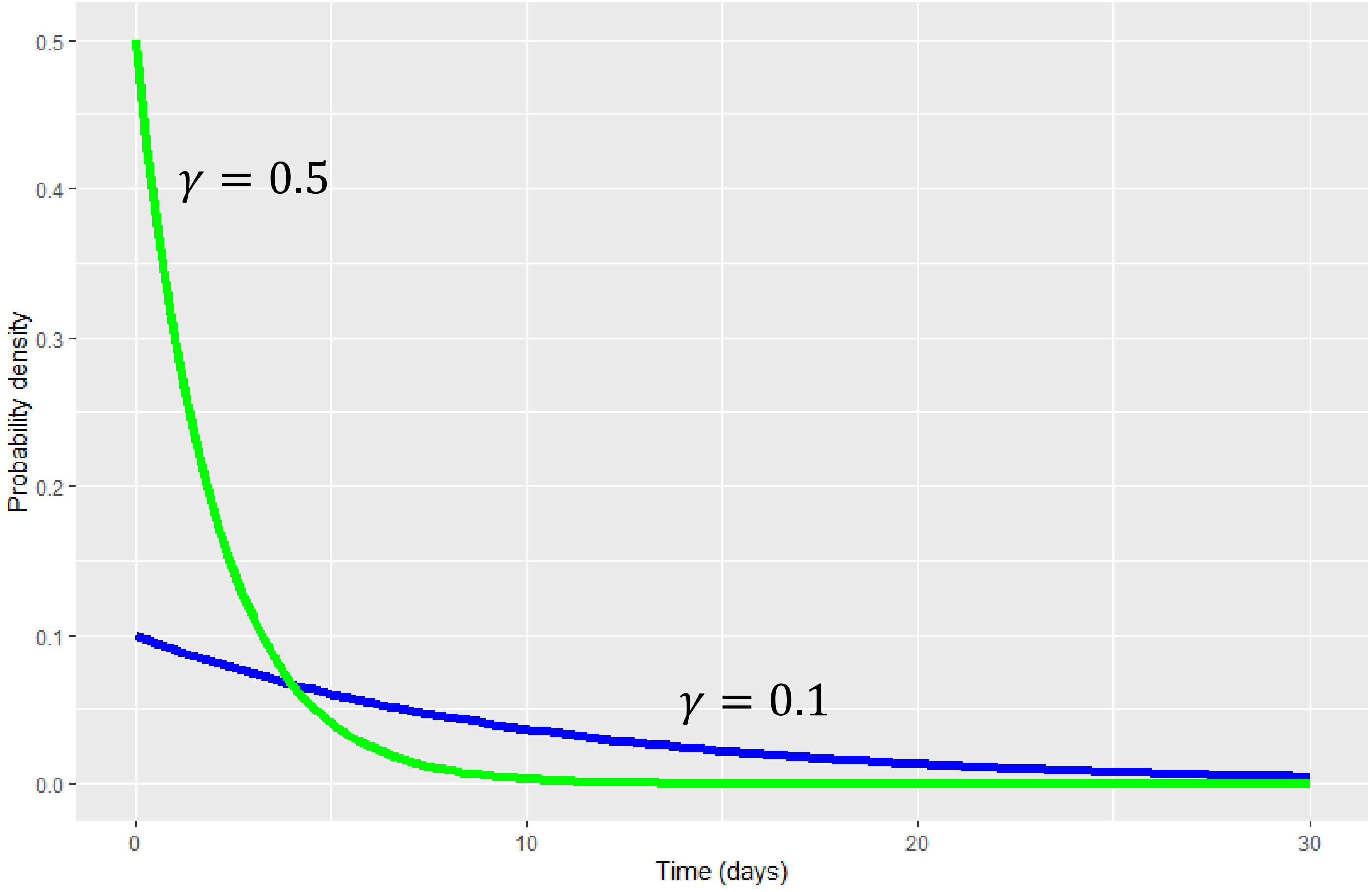


Behaviour of the exponential distribution



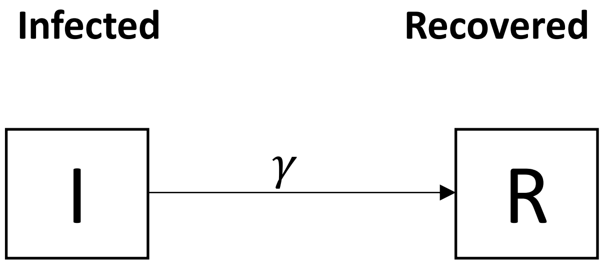
- For γ = 0.1 , we can say the mean infectious period is 10 days
- For γ = 0.5 , we can say the mean infectious period is 2 days

Behaviour of the exponential distribution



- The time spent in *I* follows an exponential distribution with exponential parameter γ
- The mean of that distribution is $1/\gamma$, which is in our case the mean infectious period (in days)
- The shorter the infectious period, the larger (quicker) the recovery rate !

Competing Hazards



- Let's add some complexity by adding a mortality rate

$$\frac{dI}{dt} = -\gamma I(t)$$

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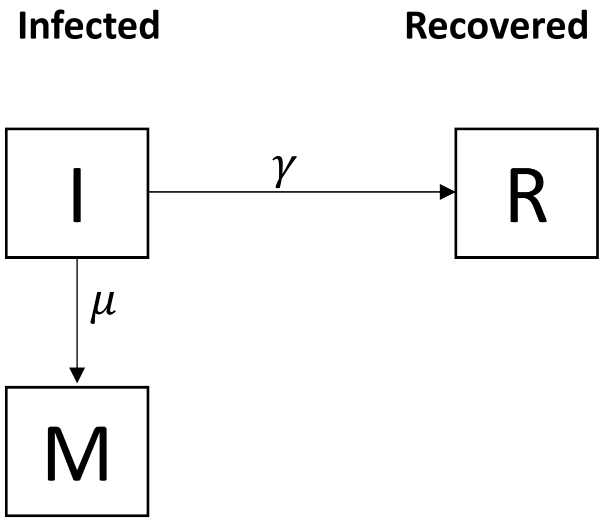
Competing Hazards

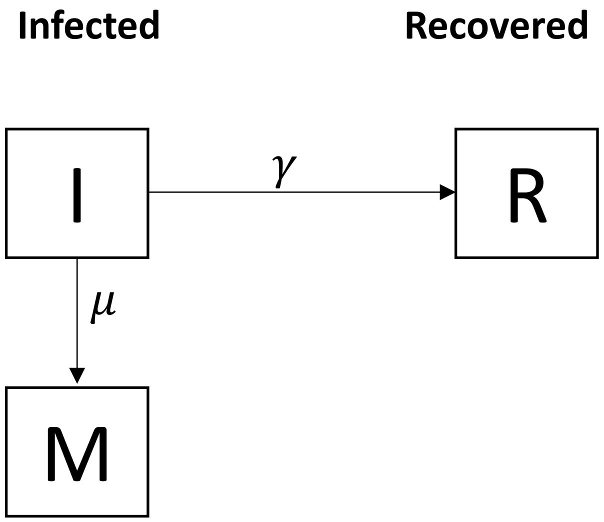


$$\frac{dI}{dt} = -(\gamma + \mu)I(t)$$

$$\frac{dR}{dt} = \gamma I(t) \qquad \frac{dM}{dt} = \mu R(t)$$

- Let's add some complexity by adding a mortality rate μ
- More than one event can flow out of *I* compartment.
- This is what we call competing hazards: $\mu \ and \ \gamma$

Competing Hazards



$$\frac{dI}{dt} = -(\gamma + \mu)I(t)$$

$$\frac{dR}{dt} = \gamma I(t) \qquad \frac{dM}{dt} = \mu R(t)$$

- if $\mu > \gamma$, means more people die before they recover (e.g, Ebola)
- This means that for a particular compartment, two hazard rates are competing
- We need to account for that when we define the value of our rates

Competing Hazards: CFR

- The case fatality rate is the proportion of people that die before they recover.
- It can be expressed as :

$$\mathrm{CFR} = \mu/(\,\gamma + \mu)$$

- Similarly, the survival rate is

$$\mathrm{survival} = \gamma/(\,\gamma + \mu)$$

Competing Hazards: estimate μ from CFR

Some algebra

$$CFR = \frac{\mu}{\mu + \gamma}$$
$$\mu = CFR(\mu + \gamma)$$
$$\mu = \mu CFR + \gamma CFR$$
$$\mu - \mu CFR = \gamma CFR$$
$$\mu(1 - CFR) = \gamma CFR$$
$$\mu = \frac{\gamma CFR}{1 - CFR}$$

This is what we need

Disease	Pathogen	Eradication status	Deaths per year (in most recent year)	Case fatality rate (if untreated)
Smallpox	Variola virus	eradicated	0	±30%
Rinderpest	Rinderpest virus	eradicated	0	up to 100%
Polio	Poliovirus	eradication under way	0	<0.5%
Guinea worm	Guinea worm (nematode)	eradication under way	not deadly	0%
Yaws	Treponema pallidum (bacterium)	eradication under way	not deadly	0%
Rabies	Lyssavirus	global elimination under way	13,289 (2016)	100%
Tuberculosis	Mycobacterium tuberculosis (bacterium)	possibly eradicable in the future	1.21 million (2016)	70%
HIV/AIDS	Human immunodeficiency virus	possibly eradicable in the future	1.03 million (2016)	up to 100%
Malaria	Plasmodium (unicellular parasite)	possibly eradicable in the future	0.72 million (2016)	±0.3%

source: https://ourworldindata.org/uploads/2018/06/Key-facts-about-eradicated-and-eradicable-diseases.png

What we should know by now

- What are compartmental models
- How we use ODEs to write these models
- What are hazard rates and how can be interpreted
- What are competing hazards and why are these important.
- What is the case fatality rate (CFR)